GRADE 12
THEOREM BOOKLET

BISHOPS
DIOCESAN COLLEGE

PAPER 1

PAPER 2

THE SUM OF ARITHMETIC SERIES

Required to Prove: $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

Proof:

$$S_n = T_1 + T_2 + T_3 + \ldots + T_{n-1} + T_n$$

$$S_n = a + (a + d) + (a + 2d) + \ldots + (l - 2d) + (1 - d) + l \ldots ①$$

$$S_n = l + (l - d) + (l - 2d) + \ldots + (a + 2d) + (a + d) + a \ldots ②$$

$$① + ② : 2S_n = \underbrace{(a + l) + (a + l) + (a + l) + \ldots + (a + l) + (a + l) + (a + l)}_{for\ n\ terms}$$

$$\therefore 2S_n = n \times (a + l)$$

$$\therefore S_n = \frac{n}{2}(a + l)$$

But $l = a + (n-1)d$

$$\therefore S_n = \frac{n}{2}(a + l) = \frac{n}{2}\left[2a + (n-1)d\right]$$

THE SUM OF GEOMETRIC SERIES

Required to Prove: $S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1} \quad for \ \ r \neq 1$

Proof:

$$S_n = T_1 + T_2 + T_3 + \ldots + T_{n-1} + T_n$$

$$S_n = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1} \ldots ①$$

$$r \times S_n = ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1} + ar^n \ldots ②$$

$$① - ② : S_n - r S_n = a - ar^n$$

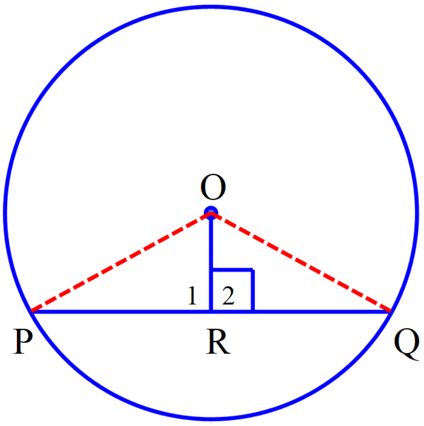
$$\therefore S_n(1 - r) = a(1 - r^n)$$

$$\therefore S_n = \frac{a(1 - r^n)}{(1 - r)} = \frac{a(r^n - 1)}{(r - 1)} \quad for \quad r \neq 1$$

THEOREM 1

The line drawn from the centre of a circle, perpendicular to a chord, bisects the chord.

Given: Circle with centre O and chord PQ. $OR \perp PQ$ with R on PQ.



Required to Prove: $PR = RQ$

Construction: OP and OQ

Proof:

In ΔOPR and ΔOQR

1) $\hat{R}_1 = \hat{R}_2 = 90^\circ$ $(OR \perp PQ)$

2) $OP = OQ$ (radii)

3) OR is common

$\therefore \Delta OPR \equiv \Delta OQR$ (RHS)

$PR = RQ$ $(\Delta OPR \equiv \Delta OQR)$

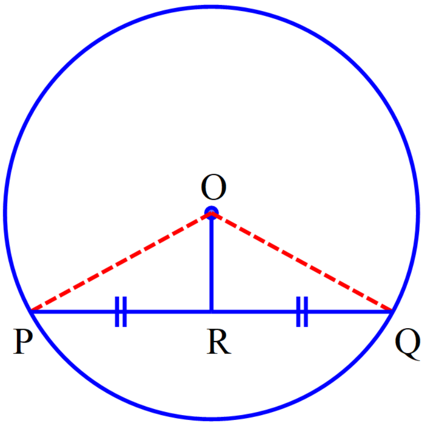
$\Rightarrow$ OR bisects PQ

The converse theorem states that the line drawn from the centre of a circle to the midpoint of a chord will be perpendicular to the chord.

THEOREM 1 (CONVERSE)

The line drawn from the centre of a circle that bisects a chord, is perpendicular to the chord

Given: Circle with centre O and chord PQ. PR = RQ with R on PQ.



Required to Prove: $OR \perp PQ$

Construction: OP and OQ

Proof:

In ΔOPR and ΔOQR

1) $OP = OQ$ (equal radii)

2) $PR = RQ$ (given)

3) OR is common

$\therefore\ \Delta OPR \equiv \Delta OQR$ (SSS)

$\therefore O\hat{R}P = O\hat{R}Q$

and $O\hat{R}P + O\hat{R}Q = 180°$ ($\angle$ on str. line)

$\therefore O\hat{R}P = O\hat{R}Q = 90°$

$\Rightarrow OR \perp PQ$

THEOREM 3

The angle subtended by an arc at the centre of the circle is double the size of the angle subtended by the same arc at any point on the circumference of the circle.

Given: A, B and C are 3 points on the circle with centre O.

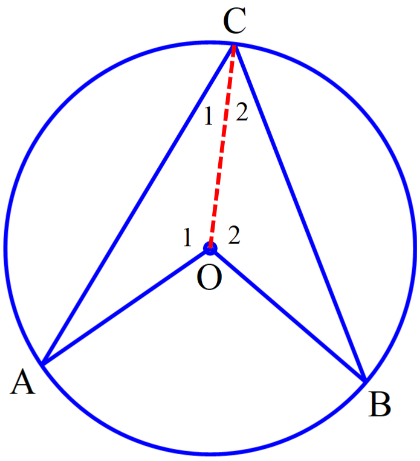


FIGURE 1

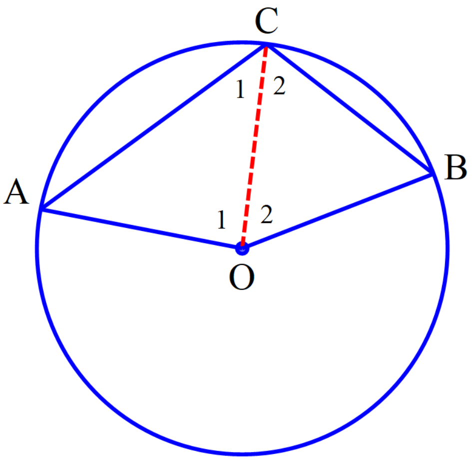


FIGURE 2

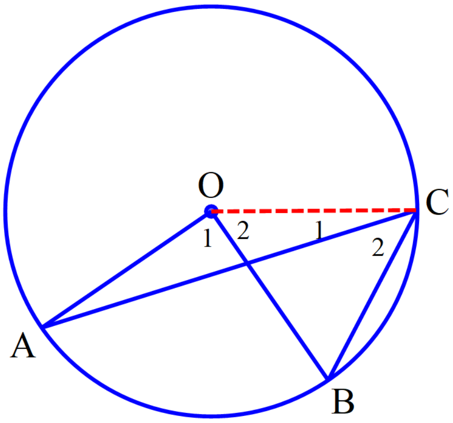


FIGURE 3

Required to Prove: In FIGURES 1 & 3 prove $A\hat{O}B = 2A\hat{C}B$

In FIGURE 2 prove **reflex** $A\hat{O}B = 2A\hat{C}B$

Construction: CO (AO = CO = OB)

Proof: FIGURE 1 and FIGURE 2

Let $\hat{C}_1 = x$ and $\hat{C}_2 = y$

$\Rightarrow \hat{A} = x$ and $\hat{B} = y$ ($\angle$'s opp = sides, radii =)

$A\hat{O}C = 180^\circ - 2x$ $(\angle \text{ sum } \Delta AOC)$

$B\hat{O}C = 180^\circ - 2y$ $(\angle \text{ sum } \Delta BOC)$

FIGURE 1: $A\hat{O}B = 2(x+y) = 2A\hat{C}B$ (sum of $\angle$'s around a point)

FIGURE 2: Reflex $A\hat{O}B = 2(x+y) = 2A\hat{C}B$ (sum of $\angle$'s around a point)

Proof: FIGURE 3

Let $\hat{C}_1 = x$ and $\hat{C}_2 = y$

$\hat{A} = x$ and $\hat{B} = x + y$ ($\angle$'s opp = sides, radii =)

$A\hat{O}C = 180^\circ - 2x$ $(\angle \text{ sum of } \Delta AOC)$

$B\hat{O}C = 180^\circ - (2x + 2y)$ $(\angle \text{ sum of } \Delta BOC)$

$A\hat{O}B = A\hat{O}C - B\hat{O}C = 2y = 2A\hat{C}B$

This theorem does not have a converse.

THEOREM 5

The opposite angles of a cyclic quadrilateral are supplementary.

Given: D, E, F and G are 4 points on the circle with centre O.

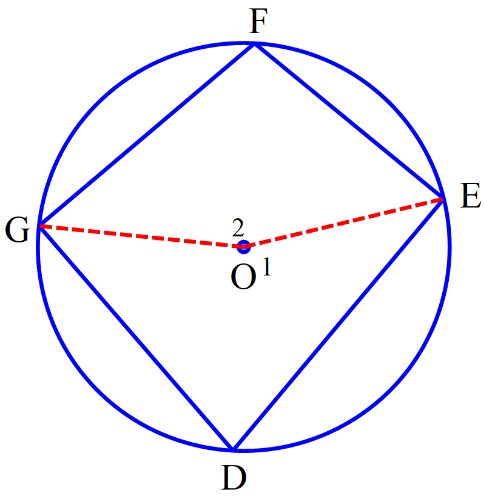


FIGURE 1

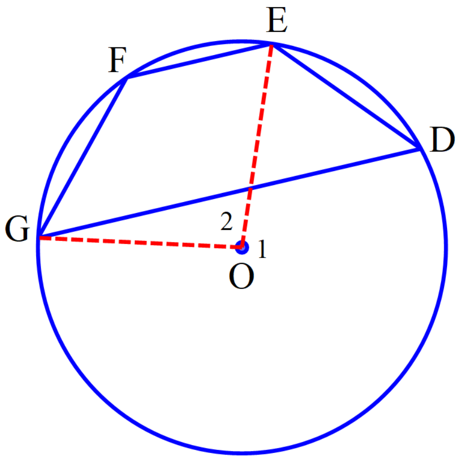


FIGURE 2

Required to Prove: $\hat{D}+\hat{F}=180^\circ$ and $D\hat{E}F+D\hat{G}F=180^\circ$

Construction: EO and GO

Proof: The proof is the same for FIGURE 1 and FIGURE 2

Let $\hat{D}=x$

$\hat{O}_2=2x$ $(\angle \text{ at centre} = 2\times\angle \text{ at circumference})$

$\hat{O}_1=360^\circ-2x$ $(\angle\text{s in a revolution})$

$\hat{F}=180^\circ-x$ $(\angle \text{ at centre} = 2\times\angle \text{ at circumference})$

$\hat{D}+\hat{F}=180^\circ$

$F\hat{E}D+F\hat{G}D=180^\circ$ $(\angle \text{ sum quad})$

The converse theorem states that a quadrilateral will be cyclic if it’s opposite angles are supplementary.

THEOREM 7

The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle subtended by the chord in the alternate segment.

Given: A, B and C are points on the circle with centre O. DA is a tangent to the circle at A.

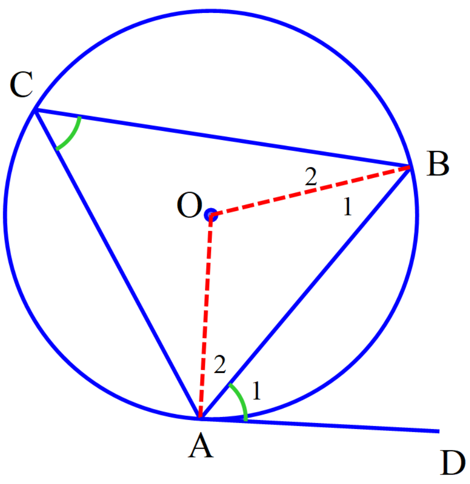


FIGURE 1

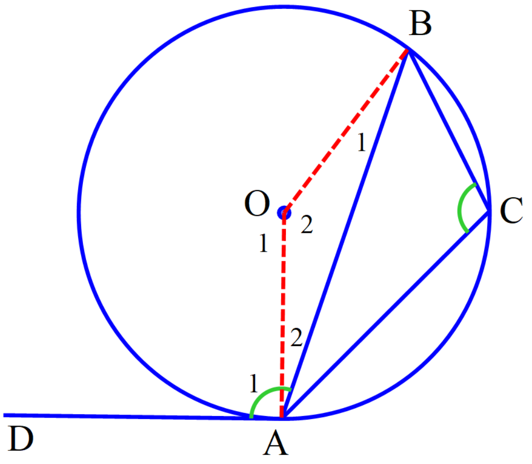


FIGURE 2

Required to Prove: In FIGURE 1, acute $B\hat{A}D = \hat{C}$

In FIGURE 2, obtuse $B\hat{A}D = \hat{C}$

Construction: OA and OB

Proof: **FIGURE 1**

Let $\hat{C} = x$

$O = 2x$ $\quad(\angle$ at centre $= 2x\angle$ at circumference$)$

$\hat{A}_2 = \hat{B}_1$ $\quad(\angle$'s opp $=$ sides, radii $=)$

$\hat{A}_2 = 90^\circ - x$ $\quad(\angle$ sum in $\triangle)$

$O\hat{A}D = 90^\circ$ $\quad(\text{rad} \perp \text{tan})$

$B\hat{A}D = x = \hat{C}$

Proof: **FIGURE 1**

Let $\hat{C} = x$

Reflex $\hat{O}_1 = 2x$ $\quad(\angle$ at centre $= 2x\angle$ at circumference$)$

$\hat{O}_2 = 360^\circ - 2x$ $\quad($sum of $\angle$'s around a point$)$

$\hat{A}_2 = \hat{B}_1$ $\quad(\angle$'s opp $=$ sides, radii $=)$

$\hat{A}_2 = x - 90^\circ$ $\quad(\angle$ sum in $\triangle)$

$O\hat{A}D = 90^\circ$ $\quad(\text{rad} \perp \text{tan})$

$B\hat{A}D = x = \hat{C}$

In FIGURE 1: Acute $B\hat{A}D = \hat{C}$

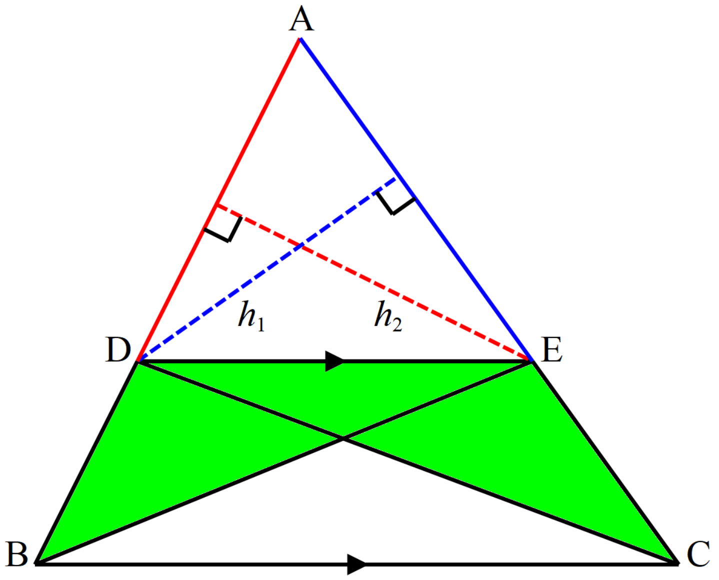
In FIGURE 2: Obtuse $B\hat{A}D = \hat{C}$

The converse theorem states that if the angle between a line and a chord equals the angle subtended by the chord in the alternate segment, then the line is a tangent to the circle.

THEOREM

A line drawn parallel to one side of a triangle divides the other two sides proportionally.

Given: ΔABC with DE || BC, such that D lies on AB and E lies on AC.



Construction: $h_1 \perp AE$ and $h_2 \perp AD$

Proof:

$$\frac{\text{Area } \Delta ADE}{\text{Area } \Delta DBE} = \frac{\frac{1}{2} AD \times h_2}{\frac{1}{2} DB \times h_2} = \frac{AD}{DB}$$ common vertex E, same height h_2

$$\frac{\text{Area } \Delta AED}{\text{Area } \Delta ECD} = \frac{\frac{1}{2} AE \times h_1}{\frac{1}{2} EC \times h_1} = \frac{AE}{EC}$$ common vertex D, same height h_1

$\text{Area } \Delta DBE = \text{Area } \Delta ECD$ common base DE, same height, DE || BC

$$\frac{\text{Area } \Delta ADE}{\text{Area } \Delta DBE} = \frac{\text{Area } \Delta AED}{\text{Area } \Delta ECD}$$ Area ΔADE is common and $\text{Area } \Delta DBE = \text{Area } \Delta ECD$

$$\Rightarrow \frac{AD}{DB} = \frac{AE}{EC}$$

Using the same sketch, we can also prove that: $\frac{AD}{AB}=\frac{AE}{AC}$

Proof:

$$\frac{\text{Area }\Delta ADE}{\text{Area }\Delta ABE}=\frac{\frac{1}{2}AD\times h_2}{\frac{1}{2}AB\times h_2}=\frac{AD}{AB}$$

common vertex E, same height h_2

$$\frac{\text{Area }\Delta AED}{\text{Area }\Delta ACD}=\frac{\frac{1}{2}AE\times h_1}{\frac{1}{2}AC\times h_1}=\frac{AE}{AC}$$

common vertex D, same height h_1

$\text{Area }\Delta DBE=\text{Area }\Delta ECD$ common base DE, same height, DE || BC

$\text{Area}\Delta ABE=\text{Area }\Delta ACD$ Area ΔADE is common and $\text{Area }\Delta DBE=\text{Area }\Delta ECD$

$$\frac{\text{Area }\Delta ADE}{\text{Area }\Delta ABE}=\frac{\text{Area }\Delta AED}{\text{Area }\Delta ACD}$$

Area ΔADE is common and

$\text{Area }\Delta ABE=\text{Area }\Delta ACD$

$$\Rightarrow\frac{AD}{AB}=\frac{AE}{AC}$$

Finally, it can also be proved that: $\frac{AB}{DB}=\frac{AC}{EC}$

Proof:

$$\frac{\text{Area }\Delta ABE}{\text{Area }\Delta DBE}=\frac{\frac{1}{2}AB\times h_2}{\frac{1}{2}DB\times h_2}=\frac{AB}{DB}$$

common vertex E, same height h_2

$$\frac{\text{Area }\Delta ACD}{\text{Area }\Delta ECD}=\frac{\frac{1}{2}AC\times h_1}{\frac{1}{2}EC\times h_1}=\frac{AC}{EC}$$

common vertex D , same height h_1

$\text{Area }\Delta DBE=\text{Area }\Delta ECD$ common base DE, same height, DE || BC

$\text{Area}\Delta ABE=\text{Area }\Delta ACD$ Area ΔADE is common and $\text{Area }\Delta DBE=\text{Area }\Delta ECD$

$$\frac{\text{Area }\Delta ABE}{\text{Area }\Delta DBE}=\frac{\text{Area }\Delta ACD}{\text{Area }\Delta ECD}$$

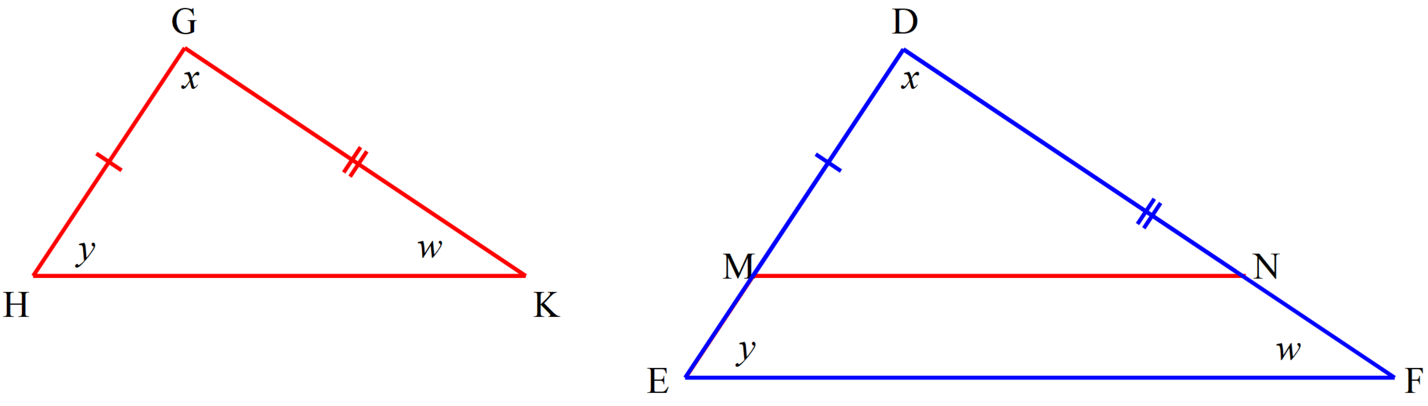
Area ΔADE is common and $\text{Area }\Delta ABE=\text{Area }\Delta ACD$

$$\Rightarrow\frac{AB}{DB}=\frac{AC}{EC}$$

THEOREM

If two triangles are equiangular, their corresponding sides are in proportion.

Given: ΔDEF and ΔGHK with $\hat{D}=\hat{G}$, $\hat{E}=\hat{H}$ and $\hat{F}=\hat{K}$



Required to Prove: $\dfrac{DE}{GH}=\dfrac{DF}{GK}=\dfrac{EF}{HK}$

Construction: On DE and DF mark points M and N so that $DM=GH$ and $DN=GK$.
Join MN.

Proof:

In ΔDMN and ΔGHK

1) $DM=GH$ (construction)

2) $DN=GK$ (construction)

3) $\hat{D}=\hat{G}$ (given)

$\therefore \Delta DMN \equiv \Delta GHK$ (SAS)

$\therefore D\hat{M}N=G\hat{H}K$ (congruency)

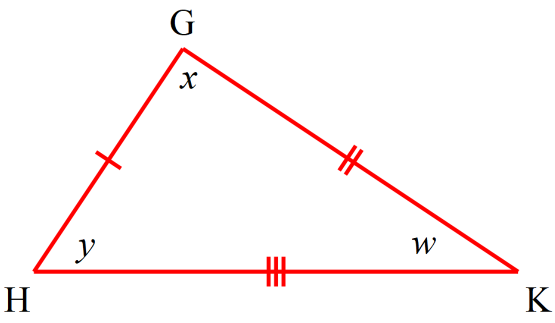
$=\hat{E}$ (given)

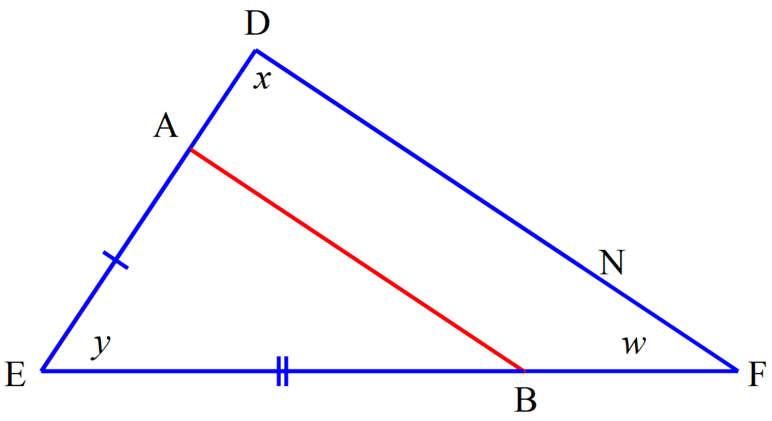
$\therefore MN \parallel EF$ (corresponding $\angle s =$)

$\therefore \dfrac{DE}{DM}=\dfrac{DF}{DN}$ (Proportionality Theorem $MN \parallel EF$)

But $DM=GH$ and $DN=GK$ (construction)

$\therefore \dfrac{DE}{GH}=\dfrac{DF}{GK}$





Similarly, by marking off points A and B on ED and EF respectively so that $\mathrm{EA} = \mathrm{HG}$ and $\mathrm{EB} = \mathrm{HK}$, it can be proved that $\dfrac{\mathrm{DE}}{\mathrm{GH}} = \dfrac{\mathrm{EF}}{\mathrm{HK}}$

$$\therefore \quad \frac{\mathrm{DE}}{\mathrm{GH}} = \frac{\mathrm{DF}}{\mathrm{GK}} = \frac{\mathrm{EF}}{\mathrm{HK}}$$

$\Rightarrow \Delta \mathrm{DEF} \,|||\, \Delta \mathrm{GHK}$ corresponding angles equal and corresponding sides in proportion

TRIGONOMETRIC COMPOUND FORMULAE

1. Derivation of $\cos(\alpha+\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$ from $\cos(\alpha-\beta)$

$$\cos(\alpha+\beta)=\cos(\alpha-(-\beta))$$

$$=\cos\alpha\cos(-\beta)+\sin\alpha\sin(-\beta)$$

But $\cos(-\beta)=\cos\beta$ and $\sin(-\beta)=-\sin\beta$

$$\therefore \cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

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2. Derivation of $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$ from $\cos(\alpha-\beta)$

$$\sin(\alpha+\beta)=\cos\left[90^\circ-(\alpha+\beta)\right]$$

$$=\cos\left[(90^\circ-\alpha)-\beta\right]$$

$$=\cos(90^\circ-\alpha)\cos\beta+\sin(90^\circ-\alpha)\sin\beta$$

But $\cos(90^\circ-\alpha)=\sin\alpha$ and $\sin(90^\circ-\alpha)=\cos\alpha$

$$\therefore \sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

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3. Derivation of $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$ from $\cos(\alpha-\beta)$

$$\sin(\alpha-\beta)=\cos\left[90^\circ-(\alpha-\beta)\right]$$

$$=\cos\left[(90^\circ-\alpha)-(-\beta)\right]$$

$$=\cos(90^\circ-\alpha)\cos(-\beta)+\sin(90^\circ-\alpha)\sin(-\beta)$$

But $\cos(90^\circ-\alpha)=\sin\alpha$; $\sin(90^\circ-\alpha)=\cos\alpha$ and $\cos(-\beta)=\cos\beta$; $\sin(-\beta)=-\sin\beta$

$$\therefore \sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

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4. Derivation of $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$ from $\sin(\alpha+\beta)$

$$\sin(\alpha-\beta)=\sin(\alpha+(-\beta))$$

$$=\sin\alpha\cos(-\beta)+\cos\alpha\sin(-\beta)$$

But $\cos(-\beta)=\cos\beta$ and $\sin(-\beta)=-\sin\beta$

$$\therefore \sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

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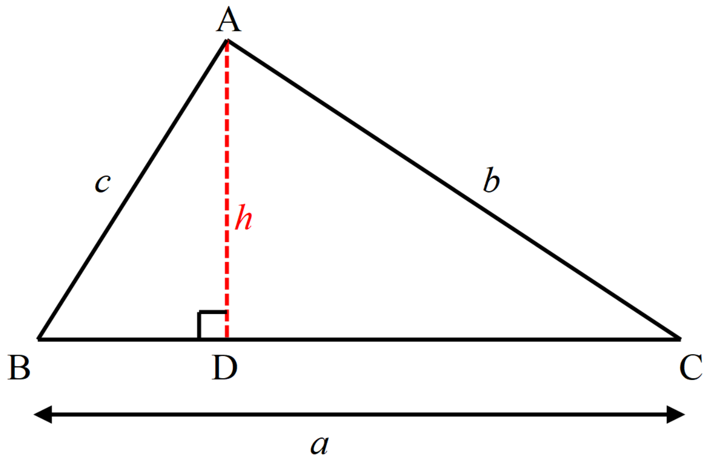
etc…

SIN RULE and its proof

Required to Prove: In any $\Delta\mathrm{ABC}: \dfrac{a}{\sin \mathrm{A}} = \dfrac{b}{\sin \mathrm{B}} = \dfrac{c}{\sin \mathrm{C}}$ *OR* $\dfrac{\sin \mathrm{A}}{a} = \dfrac{\sin \mathrm{B}}{b} = \dfrac{\sin \mathrm{C}}{c}$

Proof:



Acute-angled triangle

Let AD = h

= height of $\Delta\mathrm{ABC}$ with base BC

$\sin \mathrm{B} = \dfrac{h}{c}$ *and* $\sin \mathrm{C} = \dfrac{h}{b}$

$\therefore h = c\sin \mathrm{B}$ *and* $h = b\sin \mathrm{C}$

Equate h on both sides:

$\therefore c\sin \mathrm{B} = b\sin \mathrm{C}$

Divide both sides by bc:

$$\therefore \frac{c\sin \mathrm{B}}{bc} = \frac{b\sin \mathrm{C}}{bc}$$

$$\therefore \frac{\sin \mathrm{B}}{b} = \frac{\sin \mathrm{C}}{c}$$

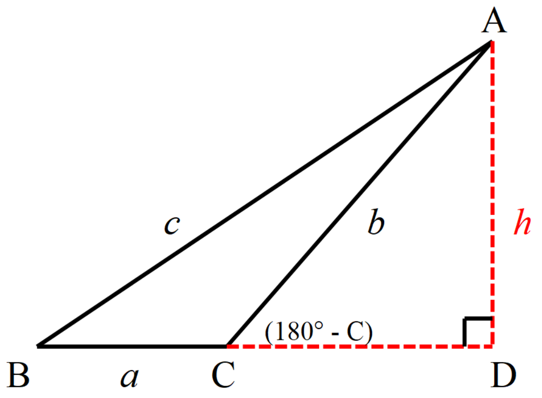
Let CE = h

= height of $\Delta\mathrm{ABC}$ with base AB

Repeat the steps above to get:

$$\frac{\sin \mathrm{B}}{b} = \frac{\sin A}{a}$$

$$\therefore \frac{\sin \mathrm{A}}{a} = \frac{\sin \mathrm{B}}{b} = \frac{\sin \mathrm{C}}{c}$$



Obtuse-angled triangle

Let AD = h

= height of $\Delta\mathrm{ABC}$ with base BC

$\sin \mathrm{B} = \dfrac{h}{c}$ *and* $\sin\left(180° - \mathrm{C}\right) = \dfrac{h}{b}$

but $\sin\left(180° - \mathrm{C}\right) = \sin \mathrm{C}$

$\therefore h = c\sin \mathrm{B}$ *and* $h = b\sin \mathrm{C}$

Equate h on both sides:

$\therefore c\sin \mathrm{B} = b\sin \mathrm{C}$

Divide both sides by bc:

$$\therefore \frac{c\sin \mathrm{B}}{bc} = \frac{b\sin \mathrm{C}}{bc}$$

$$\therefore \frac{\sin \mathrm{B}}{b} = \frac{\sin \mathrm{C}}{c}$$

Let AD = h

= height of $\Delta\mathrm{ABC}$ with base AB

Repeat the steps above to get:

$$\frac{\sin \mathrm{B}}{b} = \frac{\sin A}{a}$$

$$\therefore \frac{\sin \mathrm{A}}{a} = \frac{\sin \mathrm{B}}{b} = \frac{\sin \mathrm{C}}{c}$$

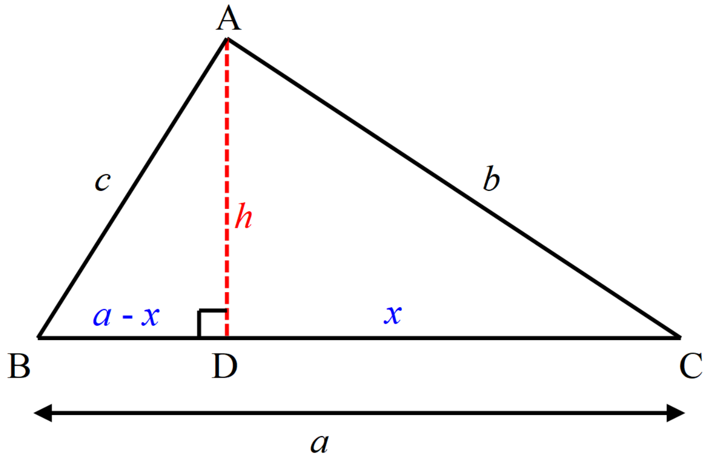
which is the same as $\dfrac{a}{\sin \mathrm{A}} = \dfrac{b}{\sin \mathrm{B}} = \dfrac{c}{\sin \mathrm{C}}$

COSINE RULE and its proof

Required to Prove: In any ΔABC:

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos \text{B}$$
$$c^2 = a^2 + b^2 - 2ab\cos \text{C}$$

Proof:



Acute-angled triangle

Let $\text{AD} = h$
$= \text{height of } \Delta\text{ABC} \text{ with base BC}$

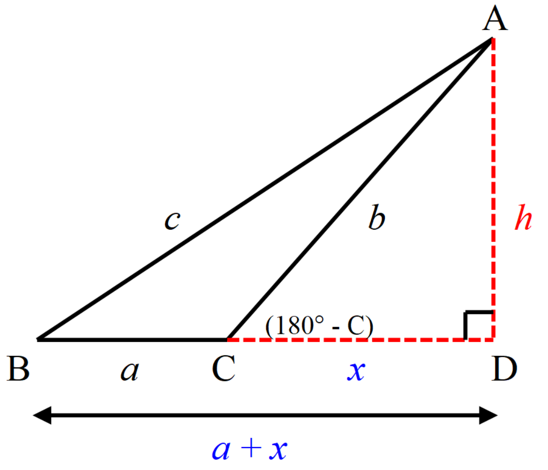
Apply Pythagoras' Theorem to ΔABD to find c^2

$$\begin{aligned} c^2 &= h^2 + (a-x)^2 \\ &= h^2 + a^2 - 2ax + x^2 \\ &= a^2 + \left(h^2 + x^2\right) - 2ax \\ &= a^2 + b^2 - 2ax \end{aligned}$$

$b^2 = h^2 + x^2 \quad (\text{Pythagoras in } \Delta\text{ADC})$

$$\frac{x}{b} = \cos \text{C}$$
$$\therefore x = b\cos \text{C}$$

$$c^2 = a^2 + b^2 - 2ab\cos \text{C}$$



Obtuse-angled triangle

Let $\text{AD} = h$
$= \text{height of } \Delta\text{ABC} \text{ with base BC}$

Apply Pythagoras' Theorem to ΔABD to find c^2

$$\begin{aligned} c^2 &= h^2 + (a+x)^2 \\ &= h^2 + a^2 + 2ax + x^2 \\ &= a^2 + \left(h^2 + x^2\right) + 2ax \\ &= a^2 + b^2 + 2ax \end{aligned}$$

$b^2 = h^2 + x^2 \quad (\text{Pythagoras in } \Delta\text{ADC})$

$$\frac{x}{b} = \cos(180° - \text{C})$$
$$\therefore x = -b\cos \text{C}$$

$\cos(180° - \text{C}) = -\cos \text{C}$

$$c^2 = a^2 + b^2 - 2ab\cos \text{C}$$

AREA RULE and its proof

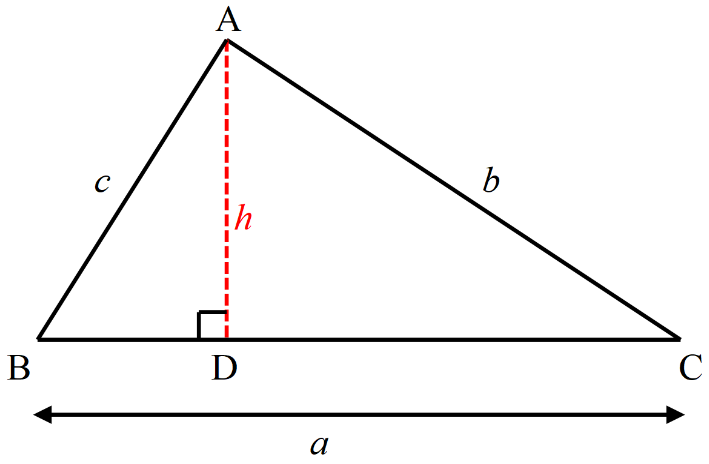
Required to Prove: In any ΔABC:

$$Area = \frac{1}{2}ab\sin C$$

$$Area = \frac{1}{2}ac\sin B$$

$$Area = \frac{1}{2}bc\sin A$$

Proof:



Acute-angled triangle

Let AD $= h$

$= \text{height of } \Delta ABC$ with base BC

$$\text{Area } \Delta ABC = \frac{1}{2}a \times h$$

$$\frac{h}{c} = \sin B \quad OR \quad \frac{h}{b} = \sin C$$

$$\therefore h = c\sin B \quad OR \quad h = b\sin C$$

Substitute h

$$\text{Area} = \frac{1}{2}ac\sin B \quad OR \quad A = \frac{1}{2}ab\sin C$$

Let CE $= h =$ height of ΔABC with base AB

Repeat the steps above to get:

$$\text{Area} = \frac{1}{2}ac\sin B \quad OR \quad A = \frac{1}{2}bc\sin A$$